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AN ASSUMED STRESS HYBRID FINITE ELEMENT METHOD FOR AN INCOMPRESSIBLE AND NEAR-INCOMPRESSIBLE MATERIAL*

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Abstract—A variational principle based on assumed stress hybrid method suitable for incompressible or near-incompressible solid is formulated for finite element analysis. One example to illustrate the use of the method is given.

1. INTRODUCTION

A FINITE element stiffness method is one of the most versatile approximate methods to analyze the stress distribution and deformation of solid continuum of complicated geometry, but finite element methods based on either the conventional assumed displacement approach or the assumed stress hybrid formulation [1, 2] may lead to serious error for nearly incompressible materials, because the influence matrix becomes nearly singular and indeed the stiffness matrix is undefined for the incompressible material. To overcome this difficulty, the mean stress must be introduced as a dependent variable in the displacement formulation [3, 4]. Herrman [5] has used this concept in the displacement model of the finite element method. In the present paper, we shall discuss how to handle this situation in the case of the assumed stress hybrid method of references 1 and 2.

2. VARIATIONAL FUNCTIONAL AND MODIFIED INFLUENCE MATRIX

The assumed stress method proposed in references 1 and 2 is based on assumed stress distribution satisfying the equilibrium equations in the interior of each discrete element and assumed compatible displacements over the interelement boundaries. This method can be derived from a variational functional

$$\pi = \sum_n \left(\int_{\partial V_n} T_i u_i \, dS - \int_{V_n} [\tfrac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} + \alpha \theta \sigma_{ii}] \, dV - \int_{S_{\sigma_n}} \bar{T}_i u_i \, dS \right) \quad (1)$$

with the stress σ_{ij} satisfying the equilibrium equations in V_n and u_i being equal to $\bar{u}_i$ over the boundaries where the displacement is prescribed. In the equation above, n is summed over all the elements, V_n is the volume of an element, ∂V_n is the boundary of V_n, S_{σ_n} is the portion of boundary where the surface traction $\bar{T}_i$ is given. α is a thermal expansion coefficient and θ is the distribution of the change in temperature.

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In the use of the finite element method, the matrix $[H]$ of equation (13) of [2] is singular when the material is incompressible. Thus it must be modified before it can be used. For simplicity, we assume that there is no distributed body force. Let us write the equilibrating stress σ_{ij} in the form

$$\sigma_{ij} = \sigma'_{ij} - p_n \delta_{ij} \tag{2}$$

for the element with volume V_n. Here σ_{ij} is represented by a polynomial, σ'_{ij} is so chosen that the constant term of σ'_{ii} is equal to zero, δ_{ij} is the Kronecker-δ and p_n is a constant in this element. In the matrix form, equation (2) may be written as,

$$\{\sigma\} = \{\sigma'\} - \{P_p\}p_n = [P]\{\beta\} - \{P_p\}p_n \tag{3}$$

where $\{\beta\}$ is a column matrix with its elements β to be the unknown coefficients and

$$\{\sigma'\} = [P]\{\beta\} \not\equiv 0 \tag{4}$$

unless $\{\beta\} \equiv 0$. The symbol $[\;]$ denotes a rectangular matrix and $\{\;\}$, a column matrix. Then,

$$\int_{V_n} C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V = \{\beta\}^T[H]\{\beta\} - 2\{\beta\}^T\{H_p\}p_n + a^2p_n^2 \tag{5}$$

where

$$\begin{aligned} \{\beta\}^T[H]\{\beta\} &= \int_{V_n} C_{ijkl}\sigma'_{ij}\sigma'_{kl}\,\mathrm{d}V \\ \{\beta\}^T\{H_p\} &= \int_{V_n} C_{iikl}\sigma'_{kl}\,\mathrm{d}V \\ a^2 &= \int_{V_n} C_{iijj}\,\mathrm{d}V. \end{aligned} \tag{6}$$

Because of this particular way of choosing σ'_{ij}, we can show that $[H]$ in the equation above is indeed positive definite. For a general compressible material this can be shown by putting $p_n = 0$ and using the fact that the strain energy is positive definite. For an incompressible material, let us write

$$\sigma'_{ij} = \sigma''_{ij} + \tfrac{1}{3}\sigma'_{kk}\,\delta_{ij} \tag{7}$$

where σ''_{ij} is, in fact, the deviatoric stress and can be written in matrix form as

$$\{\sigma''\} = [P_d]\{\beta\}.$$

Then,

$$\{\beta\}^T[H]\{\beta\} = \int_{Vn} C_{ijkl}\sigma'_{ij}\sigma'_{kl}\,\mathrm{d}V = \int_{Vn} C_{ijkl}\sigma''_{ij}\sigma''_{kl}\,\mathrm{d}V \tag{8}$$

since $C_{iikl} = C_{klii} = 0$ for incompressible material. The right hand side of equation (8) is positive unless the deviatoric stress σ''_{ij} is identically equal to zero. Now we can show that $[H]$ is positive definite by showing that $\sigma''_{ij} \not\equiv 0$ unless $\{\beta\} \equiv 0$. From the equilibrium equations, we have

$$\sigma''_{ij,j} = -\tfrac{1}{3}\sigma'_{kk,i}$$

If $\sigma''_{ij} \equiv 0$ in V_n, σ'_{kk} will be a constant. From previous assumptions, σ'_{kk} must be zero. Then by equation (7), $\sigma''_{ij} \equiv 0$ implies $\sigma'_{ij} \equiv 0$. Therefore, by equation (4), σ''_{ij} will not be identically zero in V_n unless $\{\beta\} \equiv 0$. Thus, we conclude that $[H]$ is a positive definite matrix. For a nearly incompressible material, $[H]$ will be the sum of a nonsingular matrix and a matrix linearly proportional to C_{iikl} and C_{iijj} which are small. Thus, $[H]$ will not be nearly singular.

With this modified nonsingular matrix, $[H]$ we can proceed to establish a finite element scheme.

3. A FINITE ELEMENT SCHEME

Similar to that of reference 2, the displacement on ∂V_n is expressed in terms of unknown coefficients $\{q\}$ as

$$\{u\} = [L]\{q\} \tag{9}$$

with $\{u\}$ approximately equal to the prescribed value on the surface where displacement is known. The elements of the matrix $[L]$ are in terms of polynomials. The substitution of equations (3) and (9) into equation (1) yields

$$\pi = \sum_n (\{\beta\}^T[T]\{q\} - p_n\{T_p\}^T\{q\} - \tfrac{1}{2}\{\beta\}^T[H]\{\beta\} + \{\beta\}^T\{H_p\}p_n - \tfrac{1}{2}a^2p_n^2 - \{\beta\}^T\{\theta\} + \theta_p p_n - \{S_0\}^T\{q\}) \tag{10}$$

where

$$\begin{aligned}
\{\beta\}^T[T]\{q\} &= \int_{\partial V_n} \sigma'_{ij}v_i u_j \, \mathrm{d}S \\
p_n\{T_p\}^T\{q\} &= p_n \int_{\partial V_n} v_i u_j \, \mathrm{d}S \\
\{\beta\}^T\{\theta\} &= \int_{V_n} \alpha\theta\sigma'_{ii} \, \mathrm{d}V \\
\theta_p &= \int_{V_n} \alpha\theta \, \mathrm{d}V \\
\{S_0\}^T\{q\} &= \int_{S_{\sigma n}} \bar{T}_i u_i \, \mathrm{d}S.
\end{aligned} \tag{11}$$

Here v_i is the direction cosine of the normal on ∂v_n. By varying π with respect to $\{\beta\}$, p_n and the unspecified $\{q\}$ and setting $\delta\pi = 0$, we obtain

$$[T]\{q\} - [H]\{\beta\} + \{H_p\}p_n - \{\theta\} = 0 \tag{12}$$

$$-\{T_p\}^T\{q\} + \{\beta\}^T\{H_p\} - \theta_p - a^2 p_n = 0 \tag{13}$$

for each element and

$$\sum_n (\{\beta\}^T[T] - p_n\{T_p\}^T - \{S_0\}^T)\{\delta q\} = 0. \tag{14}$$

If the material is not incompressible, in principle, from equations (12) and (13), $\{\beta\}$ and p_n can be solved in terms of $\{q\}$. But for a near-incompressible material, $\{H_p\}$ and a^2 which are linearly proportional to C_{iikl} and C_{iijj} respectively, are very small, serious numerical error may be introduced. Therefore we would only use equations (12) to solve for $\{\beta\}$ i.e.

$$\{\beta\} = [H]^{-1}([T]\{q\}+\{H_p\}p_n-\{\theta\}). \tag{15}$$

Substituting into equation (10), we get

$$\pi = \sum_n (\tfrac{1}{2}\{M\}^T[H]^{-1}\{M\}-p_n\{T_p\}^T\{q\}-\tfrac{1}{2}a^2p_n^2+\theta_p p_n-\{S_0\}^T\{q\}) \tag{16}$$

where

$$\{M\} = [T]\{q\}+\{H_p\}p_n-\{\theta\}. \tag{17}$$

For an incompressible material, since C_{iikl} and C_{iijj} are zero, equation (13) becomes

$$\{T_p\}^T\{q\} - \theta_p = 0$$

which is a condition of constraint for $\{q\}$. The equation

$$\{T_p\}^T\{q\} = \int_{\partial V_n} v_i u_i \, \mathrm{d}S$$

says that the total change of volume of the element is due to the thermal effect only. Hence $\theta_p = 0$, i.e. when there is no thermal effect, the total volume of the element is unchanged.

Equation (16) is similar to equation (18) of reference 2. If we multiply out the term $\{M\}^T[H]^{-1}\{M\}$, we will have a term

$$\{q\}^T[T]^T[H]^{-1}[T]\{q\}$$

which is the same form as that of equation (18) of reference 2, except $[H]$ and $[T]$ are modified according to equations (6) and (11) respectively. We shall call this matrix

$$[T]^T[H]^{-1}[T]$$

a modified element stiffness matrix. Similar to reference 2, equation (16) is used to derive the finite element equations to solve for $\{q\}$ and p_n and all the conclusions drawn in reference 2 will also follow.

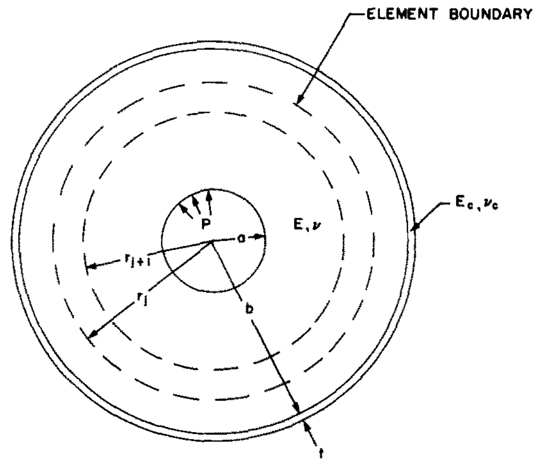
In the case of a distributed body force, equations (2) and (3) must include a particular solution, say $[P_B]\{\beta_B\}$, of the equilibrium equation, thus equation (3) becomes

$$\{\sigma\} = [P]\{\beta\}-[P_p]p_n-[P_B]\{\beta_B\}. \tag{18}$$

The rest of the equations will have to be modified accordingly. Similar to that of reference 2, the modified element stiffness matrix $[T]^T[H]^{-1}[T]$ will not change because of the distributed body force.

4. EXAMPLE

As an example to illustrate the present method, the plane strain problem of a pressurized thick-walled cylinder contained in a thin case (Fig. 1) is considered. For the case of an axial symmetric deformation of an isotropic material without thermal effect, equation (1)



$$\frac{E_c t}{Ea}\frac{1}{1-\nu_c^2} = 100, \quad \frac{b}{a} = 4$$

FIG. 1. Configuration of the thick-walled cylinder with case.

becomes

$$\pi = 2\pi\left[\sum_{n=1}^{m}\left((\sigma_r ur)\Big|_{r_{n+1}}^{r_n} - \frac{1}{2}\int_{r_{n+1}}^{r_n}\frac{1}{E}\{(1-\nu^2)(\sigma_r+\sigma_\theta)^2 - 2(1+\nu)\sigma_r\sigma_\theta\}r\,\mathrm{d}r\right) - Pr_{m+1}u(r_{m+1}) + \frac{1}{2}\frac{E_c t}{1-\nu_c^2}\frac{u^2(r_1)}{r_1}\right] \tag{19}$$

for m elements, where σ_r, σ_θ can be expressed in terms of stress function ϕ as

$$\sigma_r = \frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}, \qquad \sigma_\theta = \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}. \tag{20}$$

Within each element, ϕ is assumed to be

$$\phi = \beta_0 r + \tfrac{1}{3}\beta_1 r^3 - \tfrac{1}{2}pr^2$$

Then the matrices $[H]$, $[T]$, etc. defined in equations (10) and (16) can be easily computed, and the resulting matrix equation for the determination of the unknowns can also be obtained by the application of the variational principle.

Two cases have been computed, i.e. $\nu = 0{\cdot}5$ (incompressible) and $\nu = 0{\cdot}4995$ (nearly incompressible) with $b/a = 4$ and $E_c t/Ea\ 1/(1-\nu_c) = 100$. Results are plotted in Figs. 2 and 3. Numerical results indicate that the convergence to the exact solution is quite rapid with the increasing of the number of elements. Usually two-element approximation will give the displacement accurate up to the third digit for this simple problem. The hoop tension does not converge as fast as other quantities, but even for the case of two-element approximation, the maximum error is less than 4 per cent. The points for the hoop stress in

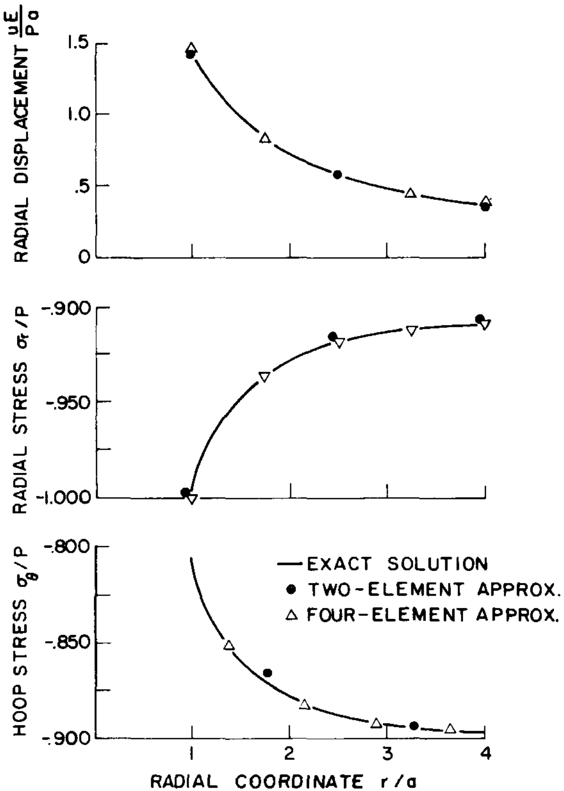


FIG. 2. Incompressible material $\nu = 0{\cdot}5$.

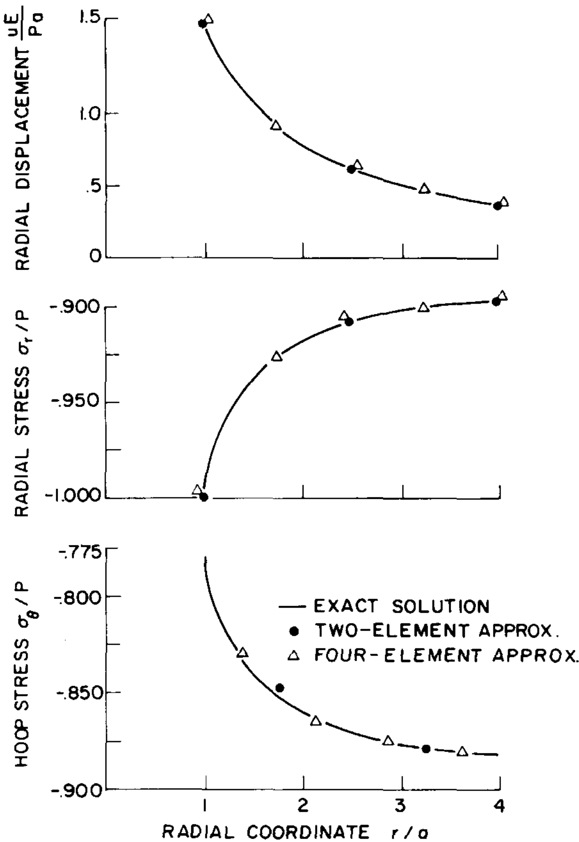


FIG. 3. Near-incompressible material $\nu = 0{\cdot}4995$.

the figures are the values of the approximate solution at the midpoint of the element. (Note that this is the same example considered in reference 5. With proper combinations of parameters, the problem depends only on three nondimensional parameters instead of five as quoted in reference 5.)

5. CONCLUSION

The singular or nearly singular difficulties faced by the assumed stress hybrid method of references 1 and 2 for the incompressible or near-incompressible materials, can be avoided by slightly modifying the influence matrix $[H]$ of references 1 and 2 by only separating the constant term of the mean stress from the assumed stress. The final system of equations remained relatively simple. Some numerical calculations for a simple problem have been made to illustrate the present method. Numerical results indicate that the convergence to exact solutions is quite rapid as the number of elements increases.

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Абстракт—Формулируется вариационный принцип основанный на смешаном методе напряжений, который является подходящим для несжимаемых и почти несжимаемых тел в задачи конечного элемента. Приводится один пример иллюстрирующий этот метод.